MULTIPLIER IDEAL SHEAVES, NEVANLINNA THEORY, AND DIOPHANTINE APPROXIMATION

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Abstract. This note states a conjecture for Nevanlinna theory or diophantine approximation, with a sheaf of ideals in place of the normal crossings divisor. This is done by using a correction term involving a multiplier ideal sheaf. This new conjecture trivially implies earlier conjectures in Nevanlinna theory or diophantine approximation, and in fact is equivalent to these conjectures. Although it does not provide anything new, it may be a more convenient formulation for some applications.

This note states a conjecture for Nevanlinna theory or diophantine approximation, with a sheaf of ideals in place of the normal crossings divisor. This is done by using a correction term involving a multiplier ideal sheaf. This new conjecture is equivalent to earlier conjectures in Nevanlinna theory or diophantine approximation, but may be a more convenient formulation for some applications. It also shows how multiplier ideal sheaves may have a role in Nevanlinna theory and diophantine approximation, and therefore may give more information on the structure of the situation.

Section 1 briefly describes multiplier ideal sheaves, and gives a variant definition specific to this situation. Section 2 describes proximity functions for sheaves of ideals, using work of Silverman and Yamanoi. Sections 3 and 4 form the heart of the paper, giving the conjectures and showing their equivalence to previous conjectures.

Throughout this paper, X is a smooth complete variety over $\mathbb{C}$ (in the case of Nevanlinna theory) or over a global field of characteristic zero (in the case of diophantine approximation).

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§1. Multiplier Ideal Sheaves

Definition 1.1. Let $\mathfrak{a}$ be a nonzero sheaf of ideals on X, and let $c \in \mathbb{R}_{\ge 0}$. Let $\mu\colon X' \to X$ be a proper birational morphism such that X' is a smooth variety and

$$\mu^*(\mathfrak{a}) = \mathscr{O}_{X'}(-F)$$

for a divisor F on X' with normal crossings support. Let $K_{X'/X}$ denote the ramification divisor of X' over X. Then the **multiplier ideal sheaf** associated to $\mathfrak{a}$ and c is the ideal sheaf

$$\mathscr{I}(\mathfrak{a}^c) = \mu_*\mathscr{O}_{X'}\bigl(K_{X'/X} - \lfloor cF \rfloor\bigr) .$$

By a theorem of Esnault and Viehweg ([**L**], Thm. 9.2.18), this definition is independent of the choice of μ.

For our purposes we need a slightly different definition.

Definition 1.2. Let $\mathfrak{a}$ and c be as above. We then define

$$\mathscr{I}^-(\mathfrak{a}^c) = \lim_{\epsilon\to 0^+} \mathscr{I}(\mathfrak{a}^{c-\epsilon}) .$$

Here we use the discrete topology on the set of ideal sheaves on X, and note that the limit exists because there are only finitely many coefficients in $\lfloor (c-\epsilon)F \rfloor$.

We also write $\mathscr{I}(\mathfrak{a}) = \mathscr{I}(\mathfrak{a}^1)$ and $\mathscr{I}^-(\mathfrak{a}) = \mathscr{I}^-(\mathfrak{a}^1)$.

Example 1.3. Let D be an (effective, reduced) normal crossings divisor on X and let $\mathfrak{a} = \mathscr{O}(-D)$. Then we can take $X' = X$, in which case $F = D$ and

$$K_{X'/X} = \lfloor (1-\epsilon)F \rfloor = 0 ,$$

so $\mathscr{I}^-(\mathfrak{a}) = \mathscr{O}_X$ (the ideal sheaf corresponding to the empty closed subscheme). More generally, if D is effective and has normal crossings support but is not necessarily reduced, then $\lfloor (1-\epsilon)F \rfloor = D - D_{\mathrm{red}}$, and therefore $\mathscr{I}^-(\mathscr{O}(-D)) = \mathscr{O}(-(D - D_{\mathrm{red}}))$.

§2. Proximity Functions for Ideal Sheaves

Silverman ([**S**], 2.2) introduced Weil functions associated to sheaves of ideals on X. By ([**S**], Thm. 2.1), there is a unique way to associate to each ideal sheaf $\mathfrak{a} \ne (0)$ of X a Weil-like function $\lambda_{\mathfrak{a}}$ on $X \setminus Y$, where Y is the closed subscheme associated to $\mathfrak{a}$, such that $\lambda_{\mathfrak{a}} = \lambda_D$ is a Weil function in the usual sense if $\mathfrak{a} = \mathscr{O}(-D)$ for some effective Cartier divisor D, and $\lambda_{\mathfrak{a}+\mathfrak{b}} = \min\{\lambda_{\mathfrak{a}}, \lambda_{\mathfrak{b}}\}$ for all nonzero ideal sheaves $\mathfrak{a}$ and $\mathfrak{b}$ of X. Here uniqueness and equality are up to addition of functions bounded by M_k-constants.

These Weil functions also satisfy the following conditions:

- They are functorial in the sense that if $f\colon X' \to X$ is a morphism of complete varieties with $f(X') \not\subseteq Y$, then $\lambda_{f^*\mathfrak{a}} = \lambda_{\mathfrak{a}} \circ f$.
- If $\mathfrak{a} \subseteq \mathfrak{b}$ are ideal sheaves on X, then $\lambda_{\mathfrak{a}} \ge \lambda_{\mathfrak{b}}$.

See also Noguchi [**N**] and Yamanoi ([**Y**], 2.2). They used similar Weil functions to define proximity functions relative to ideal sheaves. These are defined as follows. Let $f\colon \mathbb{C} \to X$ be a holomorphic curve whose image is not entirely contained in Y. Then we define the proximity function in the usual way:

$$m_f(\mathfrak{a}, r) = \int_0^{2\pi} \lambda_{\mathfrak{a}}(f(re^{i\theta}))\frac{d\theta}{2\pi}\ ,$$

and similarly in the diophantine case. Again, this agrees with $m_f(D, r)$ (up to $O(1)$) if $\mathfrak{a} = \mathscr{O}(-D)$, and satisfies the above two additional properties (again, up to $O(1)$).

§3. The Conjectures

In Nevanlinna theory, we can make the following conjecture:

Conjecture 3.1. *Let X be a nonsingular complete complex variety, let K be the canonical divisor class on X, let $\mathfrak{a} \ne (0)$ be an ideal sheaf on X, let A be a big divisor on X, and let $\epsilon > 0$. Then there is a proper Zariski-closed subset Z of X, depending only on X, $\mathfrak{a}$, A, and ϵ, such that if $f\colon \mathbb{C} \to X$ is a holomorphic curve whose image is not contained in Z, then*

$$T_{K,f}(r) + m_f(\mathfrak{a}, r) - m_f(\mathscr{I}^-(\mathfrak{a}), r) \le_{\text{exc}} \epsilon\, T_{A,f}(r) + O(1)\ .$$

Here the subscript "exc" means that the inequality holds outside of a set of r of finite Lebesgue measure.

One can also make the corresponding conjecture for algebroid functions.

The corresponding conjecture in number theory is:

Conjecture 3.2. *Let k be a global field of characteristic zero, let S be a finite set of places of k containing all archimedean places, let r be a positive integer, let X be a nonsingular complete variety over k, let K be the canonical divisor class of X, let $\mathfrak{a} \ne (0)$ be an ideal sheaf on X, let A be a big divisor on X, and let $\epsilon > 0$. Then there is a proper Zariski-closed subset Z of X, depending only on k, S, X, $\mathfrak{a}$, r, A, and ϵ, such that*

$$h_K(P) + m(\mathfrak{a}, P) - m(\mathscr{I}^-(\mathfrak{a}), P) \le \epsilon\, h_A(P) + d(P) + O(1)$$

for all $P \in (X \setminus Z)(\bar{k})$ with $[k(P) : k] \le r$.

These conjectures obviously generalize earlier conjectures in each case. Indeed, let D be an effective, reduced normal crossings divisor and let $\mathfrak{a} = \mathscr{O}(-D)$. Then $m_f(\mathscr{I}^-(\mathfrak{a}), r) = O(1)$ and $m_f(\mathfrak{a}, r) = m_f(D, r)$, and likewise in the diophantine case.

Proposition 3.3. *Conjectures 3.1 and 3.2 are equivalent to their respective special cases in which $\mathfrak{a} = \mathscr{O}(-D)$ with D as above.*

Proof. Let $\mu\colon X' \to X$, $K_{X'/X}$, and F be as in the definition of multiplier ideal sheaf, and choose $\eta > 0$ such that $\mathscr{I}^-(\mathfrak{a}) = \mathscr{I}(\mathfrak{a}^{1-\eta})$. In the Nevanlinna case, let $g\colon \mathbb{C} \to X'$ be a lifting of f; then

$$\begin{aligned}
&T_{K_X,f}(r) + m_f(\mathfrak{a},r) - m_f(\mathscr{I}^-(\mathfrak{a}),r) \\
&\qquad \le T_{K_{X'},g}(r) - m_g(K_{X'/X},r) + m_g(F,r) - m_g(-K_{X'/X} + \lfloor (1-\eta)F \rfloor, r) + O(1) \\
&\qquad = T_{K_{X'},g}(r) + m_g(F_{\mathrm{red}},r) + O(1) \\
&\qquad \le_{\mathrm{exc}} \epsilon T_{A,f}(r) + O(1) .
\end{aligned}$$

Here we use the fact that

$$\mu^*\mathscr{I}(\mathfrak{a}^{1-\eta}) = \mu^*\mu_*\mathscr{O}_{X'}(K_{X'/X} - \lfloor (1-\eta)F \rfloor) \subseteq \mathscr{O}_{X'}(K_{X'/X} - \lfloor (1-\eta)F \rfloor)$$

and therefore

$$\begin{aligned}
m_f(\mathscr{I}(\mathfrak{a}^{1-\eta}),r) &\ge m_g(\mathscr{O}_{X'}(K_{X'/X} - \lfloor (1-\eta)F \rfloor),r) + O(1) \\
&= m_g(-K_{X'/X} + \lfloor (1-\eta)F \rfloor, r) + O(1) .
\end{aligned}$$

The diophantine case is similar and is left to the reader. □

Remark 3.4. Although an arbitrary complete variety may not have a big line sheaf (or any nontrivial line sheaf) ([**F**], pp. 25–26 and p. 72), a nonsingular complete variety always does. Indeed, let U be a nonempty open affine on a nonsingular complete variety X, pick generators $x_1,\dots,x_r$ for the affine ring $\mathscr{O}_X(U)$, and let D be a Weil divisor whose support contains the polar divisors of all x_i. Then D is big.

§4. Truncated Counting Functions

Variations of the above conjectures using truncated counting functions can also be made. First of all, in Nevanlinna theory, we have:

Conjecture 4.1. *Let T be a Riemann surface, let $t\colon T \to \mathbb{C}$ be a proper surjective holomorphic map, let X be a nonsingular complete complex variety, let K be the canonical divisor class on X, let $\mathfrak{a} \ne (0)$ be a sheaf of ideals on X, let A be a big divisor on X, and let $\epsilon > 0$. Then there is a proper Zariski-closed subset Z of X, depending only on $\deg t$, X, $\mathfrak{a}$, A, and ϵ, such that for all nonconstant holomorphic curves $f\colon T \to X$ whose images are not contained in Z, the inequality*

$$N_f^{(1)}(\mathfrak{a},r) + N_{t,\mathrm{Ram}}(r) \ge_{\mathrm{exc}} T_{K,f}(r) + T_{\mathfrak{a},f}(r) - T_{\mathscr{I}^-(\mathfrak{a}),f}(r) - \epsilon T_{A,f}(r) - O(1)$$

holds.

In the diophantine case, the corresponding conjecture is:

Conjecture 4.2. *Let k be a global field of characteristic zero, let S be a finite set of places of k containing all archimedean places, let r be a positive integer, let X be a nonsingular complete variety over k, let K be the canonical divisor class of X, let $\mathfrak{a} \ne (0)$ be an ideal sheaf on X, let A be a big divisor on X, and let $\epsilon > 0$. Then there is a proper Zariski-closed subset Z of X, depending only on k, S, X, $\mathfrak{a}$, r, A, and ϵ, such that*

$$N_S^{(1)}(\mathfrak{a}, P) + d_k(P) \ge h_K(P) + h_{\mathfrak{a}}(P) - h_{\mathscr{I}^-(\mathfrak{a})}(P) - \epsilon\, h_A(P) - O(1)$$

for all $P \in (X \setminus Z)(\bar{k})$ with $[k(P) : k] \le r$.

It is not clear that the dependence of Z on $\deg t$ or r (respectively) is necessary.

In each case, if $\mathfrak{a} = \mathscr{O}(-D)$ with D an effective, reduced, normal crossings divisor, then the above conjectures reduce to conjectures that have already been posed; see [**V**] for the diophantine case.

Again, we have a converse:

Proposition 4.3. *Conjectures 4.1 and 4.2 are equivalent to their respective special cases in which $\mathfrak{a} = \mathscr{O}(-D)$ with D as above.*

Proof. In the diophantine case this follows by the same argument as before. Indeed, let $\mu\colon X' \to X$, F, and η be as before; assuming that ([**V**], Conj. 2.3) holds for F_{red} on X', we have

$$\begin{aligned}
N_S^{(1)}(\mathfrak{a}, P) + d_k(P) &= N_S^{(1)}(F_{\mathrm{red}}, P') + d_k(P') \\
&\ge h_{K_{X'}+F_{\mathrm{red}}}(P') - \epsilon\, h_{\mu^*A}(P') - O(1) \\
&= h_{K'_X}(P') - h_{K_{X'/X}}(P') + h_F(P') - h_{-K_{X'/X}+\lfloor(1-\eta)F\rfloor}(P') \\
&\qquad - \epsilon h_A(P) - O(1) \\
&\ge h_{K_X}(P) + h_{\mathfrak{a}}(P) - h_{\mathscr{I}^-(\mathfrak{a})}(P) - \epsilon h_A(P) - O(1)\,,
\end{aligned}$$

where $P' \in X'$ lies over $P \in X$. The proof in the Nevanlinna case is analogous. $\square$

References

[F] W. Fulton, *Introduction to toric varieties*, Annals of Math. Studies 131, Princeton Univ. Press, Princeton, NJ, 1993.

[L] R. Lazarsfeld, *Positivity in algebraic geometry, II*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, 49, Springer, Berlin Heidelberg New York, 2004.

[N] J. Noguchi, *Nevanlinna Theory in Several Variables and Diophantine Approximation [Japanese]*, Kyoritsu Publ., Tokyo, 2003.

[S] J. H. Silverman, *Arithmetic distance functions and height functions in diophantine geometry*, Math. Ann. **279** (1987), 193–216.

[V] P. Vojta, *A more general abc conjecture*, Intern. Math. Res. Notices **1998** (1998), 1103–1116.

[Y] K. Yamanoi, *Algebro-geometric version of Nevanlinna's lemma on logarithmic derivative and applications*, Nagoya Math. J. **173** (2004), 23–63.

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